SUTD 40.616—Special Topics in Games, Learning, and Optimisation

Lecture 16—Foundations of Variational Inequalities

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November 18, 2025

Abstract

In these notes, we study the VI problem and its applications to continuous games. We view Nash equilibria through an operator-theoretic lens, treating them as solutions of variational inequalities induced by the game's pseudo-gradient. We consider continuous games with compact, convex strategy sets and differentiable payoff functions that are concave in the player's strategies, show that such games admit at least one Nash equilibrium (NE), and characterise the set of NE as the set of solutions to a corresponding VI problem. Finally, we introduce monotone games and strongly monotone games, showing that strong monotonicity of the pseudo-gradient guarantees uniqueness of the NE.

Disclaimer. These lecture notes are a working draft and will be revised and expanded over time. They do not aim to cover the subject exhaustively; the goal is to highlight key ideas and develop some central proofs in detail. The topic is an active research area, so both the notes and our understanding of the material may evolve.

Contents

List of Abbreviations

NE	**N**ash **E**quilibrium
VI	**V**ariational **I**nequality

1 From normal-form games to continuous games

Recall that in a (finite) normal-form game between n players, each player $i \in [\![n]\!]$ has a finite set of pure strategies $\mathcal{S}_i$ and a payoff function $u_i \colon \mathcal{S} \to \mathbb{R}$, where $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ is the set of pure-strategy profiles. Normal-form games model scenarios where each player has a discrete set of actions to choose from. However, in many applications, players may have *a continuum* of actions to choose from. For example, in an auction, bidders may choose any nonnegative real number as their bid, and in a market competition, firms may choose any nonnegative production quantity. To model such scenarios, we extend the concept of normal-form games to *continuous games*, where each player's set of pure strategies is a (nonempty) subset of a Euclidean space, and each player's payoff function is a continuous function.

1.1 Continuous games

Formally, a continuous game is defined as follows.

Definition 1 (Continuous game)**.** An n-player continuous game is a tuple $\mathscr{G} \equiv \big([\![n]\!], (S_i)_{i\in[\![n]\!]}, (u_i)_{i\in[\![n]\!]}\big)$, where $\mathcal{S}_i \subseteq \mathbb{R}^{d_i}$ is the (nonempty) set of pure strategies of player $i \in [\![n]\!]$, and $u_i \colon \mathcal{S} \to \mathbb{R}$ is the continuous payoff function of player i. Here, $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \subseteq \mathbb{R}^d$ with $d = \sum_{i=1}^n d_i$.

As an example, consider a Cournot competition [1] between n firms producing a homogeneous good, where each firm $i \in [\![n]\!]$ chooses a nonnegative production quantity $q_i \in \mathbb{R}_{\geq 0}$ to maximize its profit. Let $q = (q_1, \ldots, q_n)$ be the vector of quantities and $Q(q) = \sum_{i=1}^n q_i$ be the total quantity produced. The market price $P \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a decreasing function of the total quantity, and each firm $i \in [\![n]\!]$ has a cost function $C_i \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ describing its production cost. The profit (payoff) of firm i is then given by

$$u_i(q) = q_i P\big(Q(q)\big) - C_i(q_i), \qquad \forall q \in \mathbb{R}^n_{\geq 0}. \tag{1}$$

If P and $C_1, \ldots, C_n$ are continuous functions, then each payoff function u_i is also continuous; thus, the Cournot competition is a continuous game in the sense of Definition 1, where the strategy set of each firm i is $\mathcal{S}_i = \mathbb{R}_{\geq 0}$.

Nash equilibria in continuous games. A pure-strategy Nash equilibrium (NE) [2] of a continuous game $\mathscr{G}$ is defined analogously to that of a normal-form game. In particular, a pure-strategy profile $s^\star = \big(s_1^\star, \ldots, s_n^\star\big) \in \mathcal{S}$ is an NE if no player $i \in [\![n]\!]$ can unilaterally deviate to any pure strategy $s_i \in \mathcal{S}_i$ and increase their payoff, i.e.,

$$u_i(s^\star) \geq u_i(s_i, s_{-i}^\star), \qquad \forall i \in [\![n]\!],\, s_i \in \mathcal{S}_i, \tag{2}$$

where $(s_i, s_{-i}^\star)$ denotes the strategy profile obtained by replacing the i-th component of $s^\star$ with s_i.

As an example, consider a Cournot competition between two firms A and B with cost functions $C_A(q_A) = 2q_A$ and $C_B(q_B) = 2q_B$, and market price function $P(Q) = 14 - Q$, where $Q = q_A + q_B$ is the total quantity produced. The payoff functions of firms A and B are given by (1) as follows:

$$u_A(q_A, q_B) = q_A(14 - q_A - q_B) - 2q_A = -q_A^2 - q_A q_B + 12q_A; \tag{3a}$$

$$u_B(q_A, q_B) = q_B(14 - q_A - q_B) - 2q_B = -q_B^2 - q_A q_B + 12q_B. \tag{3b}$$

Each payoff function u_i is a concave quadratic, and therefore the best-response maps of the two firms are the *singletons* $\mathrm{BR}_A(q_B) = \big(\frac{1}{2}(12 - q_B)\big)_+$ and $\mathrm{BR}_B(q_A) = \big(\frac{1}{2}(12 - q_A)\big)_+$, respectively, where $(x)_+ = \max\{0, x\}$. Then, *by symmetry*, the unique NE of this continuous game is given by $(q_A^\star, q_B^\star) = (4, 4)$, since $q_A^\star = \mathrm{BR}_A(q_B^\star)$ and $q_B^\star = \mathrm{BR}_B(q_A^\star)$. One can verify that this is indeed a NE by checking that (2) holds for both firms.

1.2 Existence of NE in continuous games

It is indeed nontrivial to establish that a continuous game admits at least one NE. The above example was simple enough to allow us to compute the NE in closed form. However, in general, we need to impose some standard assumptions on continuous games to guarantee the existence of NE. One such standard result is given by the following theorem.

Theorem 2 (Debreu [3]). *Let $\mathscr{G}$ be an n-player continuous game as in Definition 1 with compact, convex strategy sets such that the payoff functions are concave in the players' strategies, i.e., for each player $i \in [\![n]\!]$ and each $s_{-i} \in \mathcal{S}_{-i} = \prod_{j \neq i} \mathcal{S}_j$, the function $s_i \mapsto u_i(s_i, s_{-i})$ is concave. Then the game $\mathscr{G}$ admits at least one pure-strategy NE.*

Proof. For each player $i \in [\![n]\!]$, consider the best-response map $\mathrm{BR}_i \colon \mathcal{S}_{-i} \to 2^{\mathcal{S}_i}$ given by

$$\mathrm{BR}_i(s_{-i}) = \arg\max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i}), \qquad \forall s_{-i} \in \mathcal{S}_{-i}. \tag{4}$$

Because $\mathcal{S}_i$ is nonempty and compact, and u_i is continuous, by *the extreme value theorem*, it follows that $\mathrm{BR}_i(s_{-i})$ is nonempty for all $s_{-i} \in \mathcal{S}_{-i}$. Next, we show that $\mathrm{BR}_i(s_{-i})$ is also compact and convex for all $s_{-i} \in \mathcal{S}_{-i}$.

Compactness of $\mathrm{BR}_i(s_{-i})$**.** Fix player $i \in [\![n]\!]$ and $s_{-i} \in \mathcal{S}_{-i}$. Define the function $f \colon \mathcal{S}_i \to \mathbb{R}$ as $f(s_i) = u_i(s_i, s_{-i})$ for all $s_i \in \mathcal{S}_i$. Because u_i is continuous and concave in $s_i \in \mathcal{S}_i$, by assumption, it follows that f is continuous and concave.

Let

$$f^\star = \max_{s_i \in \mathcal{S}_i} f(s_i) = \max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i}), \tag{5}$$

which is guaranteed to exist since $\mathrm{BR}_i(s_{-i})$ is nonempty. Then observe that

$$\mathrm{BR}_i(s_{-i}) \equiv \mathcal{S}_i \cap f^{-1}\big(\{f^\star\}\big). \tag{6}$$

Because the preimage of a closed set under a continuous function is closed, it follows that $f^{-1}(\{f^\star\})$ is closed; thus, since $\mathcal{S}_i$ is compact, $\mathrm{BR}_i(s_{-i})$ is a closed subset of a compact set and hence compact.

Convexity of $\mathrm{BR}_i(s_{-i})$**.** Observe that

$$\mathrm{BR}_i(s_{-i}) \equiv \big\{ s_i \in \mathcal{S}_i \;\big|\; f(s_i) = f^\star \big\} \equiv \big\{ s_i \in \mathcal{S}_i \;\big|\; f(s_i) \geq f^\star \big\}. \tag{7}$$

Because the super-level sets of a concave function over a convex set are also convex, it follows that $\mathrm{BR}_i(s_{-i})$ is convex. Consequently, $\mathrm{BR}_i(s_{-i})$ is nonempty, compact, and convex for all $s_{-i} \in \mathcal{S}_{-i}$.

Upper hemicontinuity of BR_i**.** Next, we show that the best-response map $\mathrm{BR}_i \colon \mathcal{S}_{-i} \to 2^{\mathcal{S}_i}$ is upper hemicontinuous for each player $i \in [\![n]\!]$. Fix player $i \in [\![n]\!]$. Let $(s_{-i}^k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{S}_{-i}$ converging to some $s_{-i} \in \mathcal{S}_{-i}$, and let $(s_i^k)_{k \in \mathbb{N}}$ be a sequence such that $s_i^k \in \mathrm{BR}_i(s_{-i}^k)$ for all $k \in \mathbb{N}$ and $s_i^k \to s_i \in \mathcal{S}_i$ as $k \to \infty$. We need to show that $s_i \in \mathrm{BR}_i(s_{-i})$.

Assume for contradiction that $s_i \notin \mathrm{BR}_i(s_{-i})$. Since $\mathcal{S}_i$ is compact (and hence closed), and $s_i^k \in \mathcal{S}_i$ with $s_i^k \to s_i$, it follows that $s_i \in \mathcal{S}_i$. Then since $\mathrm{BR}_i(s_{-i})$ is nonempty, there exists $y_i \in \mathrm{BR}_i(s_{-i})$ such that

$$u_i(y_i, s_{-i}) > u_i(s_i, s_{-i}), \tag{8}$$

Consequently, by the continuity of u_i, there exists $K \in \mathbb{N}$ such that

$$u_i(y_i, s_{-i}^k) > u_i(s_i^k, s_{-i}^k), \qquad \forall k \geq K. \tag{9}$$

But then, since $s_i^k \in \mathrm{BR}_i(s_{-i}^k)$ for all $k \in \mathbb{N}$, it also holds that

$$u_i(s_i^k, s_{-i}^k) \geq u_i(y_i, s_{-i}^k), \qquad \forall k \in \mathbb{N}, \tag{10}$$

which is a contradiction. Thus, it must be that $s_i \in \mathrm{BR}_i(s_{-i})$, and therefore, BR_i is upper hemicontinuous.

Existence of an NE. Define the best-response correspondence $\mathrm{BR}\colon \mathcal{S} \to 2^{\mathcal{S}}$ as

$$\mathrm{BR}(s) = \mathrm{BR}_1(s_{-1}) \times \cdots \times \mathrm{BR}_n(s_{-n}), \qquad \forall s \in \mathcal{S}. \tag{11}$$

Because each BR_i is upper hemicontinuous with nonempty, compact, and convex values for all $i \in [\![n]\!]$, it follows that BR is also upper hemicontinuous with nonempty, compact, and convex values. Thus, by *Kakutani's fixed-point theorem*, there exists at least one fixed point $s^\star \in \mathcal{S}$ such that $s^\star \in \mathrm{BR}(s^\star)$. By definition of BR, this means that $s_i^\star \in \mathrm{BR}_i(s_{-i}^\star)$ for all $i \in [\![n]\!]$, which is equivalent to (2). Therefore, $s^\star$ is a NE of the game $\mathscr{G}$. □

Throughout these notes, we assume that the conditions of Theorem 2 hold; that is, we consider continuous games with *compact and convex strategy sets* and *payoff functions that are concave in the player's strategies*. Furthermore, we assume that the payoff functions are also *differentiable*.

2 The VI problem

In this section, we introduce VI problems, which *in a certain sense* generalize convex optimization problems. Formally, the VI problem[1] is defined as follows.

Definition 3 (Stampacchia VI problem)**.** Given a nonempty set $\mathcal{X} \subseteq \mathbb{R}^d$ and an operator $F\colon \mathcal{X} \to \mathbb{R}^d$, the VI problem $(F, \mathcal{X})$ is to

$$\text{find } x^\star \in \mathcal{X} \text{ such that } \big\langle F(x^\star),\, x - x^\star \big\rangle \geq 0 \text{ for all } x \in \mathcal{X}. \tag{12}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$. Any $x^\star \in \mathcal{X}$ satisfying (12) is called a solution of the VI problem $(F, \mathcal{X})$.

Various interesting problems can be formulated as VI problems. Let us begin with convex optimization problems.

Convex optimization as a special case of VI problems. Consider a nonempty convex set $\mathcal{X} \subseteq \mathbb{R}^d$ and an operator $F\colon \mathcal{X} \to \mathbb{R}^d$. Then the VI problem $(F, \mathcal{X})$ consists of finding a vector $x^\star \in \mathcal{X}$ such that $F(x^\star)$ forms *a non-obtuse angle with any feasible displacement* $x - x^\star$ *within* $\mathcal{X}$; equivalently,

$$F(x^\star) \in \mathcal{N}_{\mathcal{X}}(x^\star), \tag{13}$$

where

$$\mathcal{N}_{\mathcal{X}}(x^\star) = \left\{ v \in \mathbb{R}^d \,\middle|\, \langle v,\, x - x^\star \rangle \geq 0,\, \forall x \in \mathcal{X} \right\} \tag{Normal cone}$$

is the *normal cone* to the set $\mathcal{X}$ at the point $x^\star$.

Observe that (13) is a generalization of the first-order optimality condition for convex optimization problems. In particular, consider the convex optimization problem

$$\min_{x \in \mathcal{X}} f(x), \tag{14}$$

where $f\colon \mathcal{X} \to \mathbb{R}$ is a convex function, and let $F = \nabla f\colon \mathcal{X} \to \mathbb{R}^d$ be the gradient of f. Then a vector $x^\star \in \mathcal{X}$ is a solution of (14) if and only if it satisfies the first-order optimality condition $\nabla f(x^\star) \in \mathcal{N}_{\mathcal{X}}(x^\star)$; equivalently, $x^\star$ is a solution of the VI problem $(F, \mathcal{X})$.

[1] The Stampacchia VI problem was first introduced by Stampacchia [4] in the context of partial differential equations.

2.1 Relation between continuous games and VI problem

Variational inequalities (VIs) provide a powerful framework to characterise NE of continuous games as solutions of suitable VI problems. Let $\mathscr{G}$ be an n-player continuous game as in Definition 1, where each payoff function $u_i\colon \mathcal{S} \to \mathbb{R}$ is differentiable. We define the *pseudo-gradient* $F\colon \mathcal{S} \to \mathbb{R}^d$ of the game $\mathscr{G}$ as the operator whose i-th block component is given by the negative gradient of the i-th player's payoff function, i.e.,

$$F(s) \stackrel{\text{def}}{=} \begin{pmatrix} -\nabla_{s_1} u_1(s) \\ -\nabla_{s_2} u_2(s) \\ \vdots \\ -\nabla_{s_n} u_n(s) \end{pmatrix}, \qquad \forall s \in \mathcal{S}. \tag{Pseudo-gradient}$$

Then, under the conditions of Theorem 2, the NE of the continuous game $\mathscr{G}$ can be characterized as the solutions of the VI problem $(F, \mathcal{S})$. Formally, we have the following theorem.

Theorem 4 (Bensoussan [5])**.** *Let $\mathscr{G}$ be a continuous game with convex strategy sets and differentiable payoff functions that are concave in the player's strategies. Let $F\colon \mathcal{S} \to \mathbb{R}^d$ be the pseudo-gradient of $\mathscr{G}$. Then a pure-strategy profile $s^\star \in \mathcal{S}$ is an NE of $\mathscr{G}$ if and only if it is a solution of the VI problem $(F, \mathcal{S})$.*

Proof. We prove each direction of the equivalence separately.

The forward direction. Suppose that $s^\star \in \mathcal{S}$ is an NE of the game $\mathscr{G}$. Fix a player $i \in [\![n]\!]$. Since $\mathcal{S}_i$ is convex, we have that $s_i^\star + t(s_i - s_i^\star) \in \mathcal{S}_i$ for all $t \in [0,1]$. Next, fix a strategy profile $s_i \in \mathcal{S}_i$ and define $f\colon [0,1] \to \mathbb{R}$ as

$$f(t) = u_i\big(s_i^\star + t(s_i - s_i^\star), s_{-i}^\star\big), \qquad \forall t \in [0,1]. \tag{15}$$

Since $s_i^\star + t(s_i - s_i^\star) \in \mathcal{S}_i$ for all $t \in [0,1]$, it follows from the best-response condition in (2) that

$$f(0) = u_i(s^\star) \geq u_i(s_i^\star + t(s_i - s_i^\star), s_{-i}^\star) = f(t), \qquad \forall t \in [0,1]. \tag{16}$$

Moreover, since u_i is differentiable, we have, by the chain rule, that

$$\big\langle \nabla_{s_i} u_i(s^\star),\, s_i - s_i^\star \big\rangle = f'(0) = \lim_{t \to 0^+} \frac{f(t) - f(0)}{t} \leq 0. \tag{17}$$

Since $i \in [\![n]\!]$ and $s_i \in \mathcal{S}_i$ are arbitrary, by negating and summing over all players $i \in [\![n]\!]$, we obtain

$$\big\langle F(s^\star),\, s - s^\star \big\rangle = \sum_{i=1}^{n} \big\langle -\nabla_{s_i} u_i(s^\star),\, s_i - s_i^\star \big\rangle \geq 0, \qquad \forall s \in \mathcal{S}. \tag{18}$$

Thus, by Definition 3, $s^\star$ is a solution of the VI problem $(F, \mathcal{S})$.

The converse direction. Suppose that $s^\star \in \mathcal{S}$ is a solution of the VI problem $(F, \mathcal{S})$. Then, by Definition 3, we have

$$\big\langle F(s^\star),\, s - s^\star \big\rangle \geq 0, \qquad \forall s \in \mathcal{S}. \tag{19}$$

Fix a player $i \in [\![n]\!]$ and $s_{-i} = s_{-i}^\star$.

$$\big\langle -\nabla_{s_i} u_i(s^\star),\, s_i - s_i^\star \big\rangle = \big\langle F(s^\star),\, s - s^\star \big\rangle \geq 0, \qquad \forall s_i \in \mathcal{S}_i. \tag{20}$$

Furthermore, since $i \in [\![n]\!]$ is arbitrary, the above holds for all players $i \in [\![n]\!]$ and all $s_i \in \mathcal{S}_i$.

Finally, since u_i is concave in $s_i \in \mathcal{S}_i$ for each player $i \in [\![n]\!]$, we have, by the first-order concavity condition, that

$$u_i(s^\star) \geq u_i(s_i, s_{-i}^\star) + \big\langle -\nabla_{s_i} u_i(s^\star),\, s_i - s_i^\star \big\rangle \geq u_i(s_i, s_{-i}^\star) \qquad \forall i \in [\![n]\!], \forall s_i \in \mathcal{S}_i. \tag{21}$$

This is the best-response condition for player i as given in (2); thus, $s^\star$ is an NE of the game $\mathscr{G}$.

□

Note that in the forward direction of the proof of Theorem 4, we *did not use* the concavity of the payoff functions in the player's strategies. Thus, any NE of a continuous game with differentiable payoff functions is a solution of the corresponding VI problem, even if the payoff functions are not concave in the player's strategies.

3 Monotone operators

We saw that the set of NE of an n-player continuous game with compact, convex strategy sets and differentiable payoff functions that are concave in the player's strategies is nonempty (cf. Theorem 2) and coincides with the solutions of a corresponding VI problem (cf. Theorem 4). For brevity, in the remainder of this section, we are going to refer to such games as *differentiable concave games*.

In this section, we study sufficient conditions on the pseudo-gradients of differentiable concave games that guarantee the uniqueness of the solution of the corresponding VI problems, and thus *the uniqueness of the NE of those games*. We begin by introducing the notion of a monotone operator.

Definition 5 (Monotone operator)**.** Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a nonempty set. An operator $F\colon \mathcal{X} \to \mathbb{R}^d$ is said to be monotone if

$$\langle F(x) - F(y),\, x - y\rangle \geq 0, \qquad \forall x, y \in \mathcal{X}. \tag{22}$$

A differentiable concave game $\mathscr{G}$ with a monotone pseudo-gradient $F\colon \mathcal{S} \to \mathbb{R}^d$ is called a monotone game.

We have already seen plenty of examples of monotone games. For instance, any differentiable zero-sum game is a monotone game, including any zero-sum normal-form game. Indeed, consider a game of rock-paper-scissors between two players, A and B, given by the payoff matrix (of A):

$$\mathbf{A} = \begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & 1 & -1 \\ P & -1 & 0 & 1 \\ S & 1 & -1 & 0 \end{array} \tag{23}$$

where R, P, and S denote the pure strategies rock, paper, and scissors, respectively. The payoff functions of players A and B in the mixed extension of the above normal-form game are given by

$$u_A(x_A, x_B) = -u_B(x_A, x_B) = x_A^\mathsf{T} \mathbf{A} x_B, \qquad \forall x_A, x_B \in \Delta_3 \tag{24}$$

where $\Delta_3 \subseteq \mathbb{R}^3$ is the 2-dimensional probability simplex. Therefore, the pseudo-gradient $F\colon \Delta_3 \times \Delta_3 \to \mathbb{R}^6$ of the game is given by

$$F(x_A, x_B) = \begin{pmatrix} -\nabla_{x_A} u_A(x_A, x_B) \\ -\nabla_{x_B} u_B(x_A, x_B) \end{pmatrix} = \begin{pmatrix} -\mathbf{A} x_B \\ \mathbf{A}^\mathsf{T} x_A \end{pmatrix}, \qquad \forall x_A, x_B \in \Delta_3. \tag{25}$$

Note that, since $\mathscr{G}$ is zero-sum, we have $\mathbf{A} = -\mathbf{A}^\mathsf{T}$; thus, for all $(x_A, x_B), (y_A, y_B) \in \Delta_3 \times \Delta_3$, we have

$$\begin{aligned} \left\langle F(x_A, x_B) - F(y_A, y_B),\, \begin{pmatrix} x_A \\ x_B \end{pmatrix} - \begin{pmatrix} y_A \\ y_B \end{pmatrix} \right\rangle &= -(x_A - y_A)^\mathsf{T} \mathbf{A} (x_B - y_B) \\ &\quad + (x_B - y_B)^\mathsf{T} \mathbf{A}^\mathsf{T} (x_A - y_A) \\ &= 0. \end{aligned} \tag{26}$$

Hence, by Definition 5, F is monotone, and thus, the mixed extension of $\mathscr{G}$ is a monotone game.

3.1 Strongly monotone operators

As is the case with convexity, we can define a stronger version of monotonicity, called *strong monotonicity*, which in turn gives rise to the notions of *strongly monotone operators* and *strongly monotone games*.

Definition 6 (Strongly monotone operator)**.** Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a nonempty set. An operator $F\colon \mathcal{X} \to \mathbb{R}^d$ is said to be μ-strongly monotone if

$$\langle F(x) - F(y),\, x - y \rangle \geq \mu \cdot \|x - y\|^2, \qquad \forall x, y \in \mathcal{X} \tag{27}$$

for some constant $\mu > 0$. A differentiable concave game $\mathscr{G}$ with a strongly monotone pseudo-gradient $F\colon \mathcal{S} \to \mathbb{R}^d$ is called a strongly monotone game.

The mixed extension of any two-player zero-sum normal-form game is monotone but not strongly monotone. However, we can construct simple examples of strongly monotone games by adding a strongly convex regularization term to the payoff functions of each player in a monotone game. For example, consider again the game of rock-paper-scissors as in (23) and (24). We can modify the payoff functions of players A and B by adding the strongly convex regularization term $\frac{\mu}{2}\|x_i\|^2$, for some $\mu > 0$ and for each $i \in \{A, B\}$. Thus, the new payoff functions become

$$\begin{aligned} \tilde{u}_A(x_A, x_B) &= u_A(x_A, x_B) - \tfrac{\mu}{2}\|x_A\|^2, \qquad \forall x_A, x_B \in \Delta_3; \\ \tilde{u}_B(x_A, x_B) &= u_B(x_A, x_B) - \tfrac{\mu}{2}\|x_B\|^2, \qquad \forall x_A, x_B \in \Delta_3. \end{aligned} \tag{28}$$

In this case, the pseudo-gradient $\tilde{F}\colon \Delta_3 \times \Delta_3 \to \mathbb{R}^6$ of the modified game is given by

$$\tilde{F}(x_A, x_B) = \begin{pmatrix} -\nabla_{x_A} \tilde{u}_A(x_A, x_B) \\ -\nabla_{x_B} \tilde{u}_B(x_A, x_B) \end{pmatrix} = \begin{pmatrix} -\mathbf{A} x_B + \mu\, x_A \\ \mathbf{A}^\top x_A + \mu\, x_B \end{pmatrix}, \qquad \forall x_A, x_B \in \Delta_3. \tag{29}$$

Therefore, for all (x_A, x_B), $(y_A, y_B) \in \Delta_3 \times \Delta_3$, we have

$$\begin{aligned} \left\langle \tilde{F}(x_A, x_B) - \tilde{F}(y_A, y_B),\, \begin{pmatrix} x_A \\ x_B \end{pmatrix} - \begin{pmatrix} y_A \\ y_B \end{pmatrix} \right\rangle &= \mu \cdot \|x_A - y_A\|^2 + \mu \cdot \|x_B - y_B\|^2 \\ &= \mu \cdot \left\| \begin{pmatrix} x_A \\ x_B \end{pmatrix} - \begin{pmatrix} y_A \\ y_B \end{pmatrix} \right\|^2. \end{aligned} \tag{30}$$

Hence, by Definition 6, $\tilde{F}$ is μ-strongly monotone, and thus, the modified game is a strongly monotone game.

Under strong monotonicity of the pseudo-gradient, we can guarantee the uniqueness of the solution of the corresponding VI problem, and thus, the uniqueness of the NE of the game. In particular, we have the following theorem.

Theorem 7 (Rosen [6])**.** *Let $\mathscr{G}$ be an n-player μ-strongly monotone game for some $\mu > 0$. Then the game $\mathscr{G}$ admits a unique pure-strategy NE.*

Proof. Existence of at least one NE follows directly from Theorem 2. We prove uniqueness by contradiction. Suppose that there exist two distinct NE $s^\star$ and $\bar{s}^\star$ of the game $\mathscr{G}$. Then, by Theorem 4, both $s^\star$ and $\bar{s}^\star$ are solutions of the VI problem $(F, \mathcal{S})$, where $F\colon \mathcal{S} \to \mathbb{R}^d$ is the pseudo-gradient of $\mathscr{G}$. Thus, by the definition of the VI problem in (12), we have

$$\langle F(s^\star),\, s - s^\star \rangle \geq 0, \quad \text{and} \quad \langle F(\bar{s}^\star),\, s - \bar{s}^\star \rangle \geq 0, \qquad \forall s \in \mathcal{S}. \tag{31}$$

In particular, by substituting $s = \bar{s}^\star$ in the first inequality and $s = s^\star$ in the second inequality, we obtain

$$\langle F(s^\star),\, \bar{s}^\star - s^\star \rangle \geq 0; \quad \text{and} \quad \langle F(\bar{s}^\star),\, s^\star - \bar{s}^\star \rangle \geq 0. \tag{32}$$

Summing both inequalities yields

$$\langle F(s^\star) - F(\bar{s}^\star),\, s^\star - \bar{s}^\star \rangle \leq 0. \tag{33}$$

Moreover, since F is μ-strongly monotone for some $\mu > 0$, we have, by (27), that

$$\mu \cdot \|s^\star - \bar{s}^\star\|^2 \leq \langle F(s^\star) - F(\bar{s}^\star),\, s^\star - \bar{s}^\star \rangle \leq 0. \tag{34}$$

Because $\|\cdot\|$ is a norm, it follows that

$$\mu \cdot \|s^\star - \bar{s}^\star\|^2 = 0 \overset{\mu>0}{\Longrightarrow} \|s^\star - \bar{s}^\star\|^2 = 0 \implies s^\star = \bar{s}^\star; \tag{35}$$

this contradicts our initial assumption that $s^\star \neq \bar{s}^\star$. Therefore, the game $\mathscr{G}$ admits at most one NE, which concludes the proof. $\square$

In this section, we have introduced the notions of monotone operator and strongly monotone operator, which in turn give rise to the notions of monotone game and strongly monotone game, respectively. We have seen that strong monotonicity of the pseudo-gradient of a differentiable concave game guarantees the uniqueness of the NE of the game. At this point, we have all the machinery required to study learning in monotone games.

References

[1] Antoine Augustin Cournot. *Recherches sur les principes mathématiques de la théorie des richesses*. Louis Hachette, 1838.

[2] John Forbes Nash Jr. "Non-Cooperative Games." In: *Annals of Mathematics* 54.2 (1951), pp. 286–295. ISSN: 0003-486X. DOI: 10.2307/1969529.

[3] Gérard Debreu. "A Social Equilibrium Existence Theorem." In: *Proceedings of the National Academy of Sciences* 38.10 (1952), pp. 886–893. ISSN: 1557-7317. DOI: 10.1073/pnas.38.10.886.

[4] Guido Stampacchia. "Formes bilineaires coercitives sur les ensembles convexes." In: *Comptes rendus hebdomadaires des séances de l'Académie des sciences* 258 (1964), pp. 4413–4416. ISSN: 0001-4036.

[5] Alain Bensoussan. "Points de Nash dans le cas de fonctionnelles quadratiques et jeux différentiels linéaires à N personnes." In: *SIAM Journal on Control and Optimization* 12 (1974), pp. 460–499. ISSN: 1095-7138. DOI: 10.1137/0312037.

[6] Judah Ben Rosen. "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games." In: *Econometrica* 33.3 (1965), pp. 520–534. ISSN: 1468-0262. DOI: 10.2307/1911749.

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